114-E FINAL EXAMINATION

NAME: ______________________________

Prof. Ghrist : Spring 2019

INSTRUCTIONS:

No book, calculator, or formula sheet.

Use a writing utensil and logic.

Show all of your work
Explain yourself clearly to receive partial credit. I recommend you attempt all the problems for partial credit.

These questions are written carefully: **do not ask for clarification during the exam.** If anything is unclear, use your best judgement and explain yourself. We will grade accordingly. If we have made a mistake, we'll fix it in the grading. Relax…

Cheating, or the appearance of cheating, will be dealt with severely. By placing your name on this page, you agree to abide by the rules.

Stay calm. All of these problems are doable. You can make it.

Best wishes!

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PROBLEM 1: Consider the following vectors in $\mathbb{R}^3$:

$$\boldsymbol{t} = \begin{pmatrix} 7 \\ 0 \\ -2 \end{pmatrix} \quad \boldsymbol{u} = \begin{pmatrix} -5 \\ 2 \\ 15 \end{pmatrix} \quad \boldsymbol{v} = \begin{pmatrix} -4 \\ 1 \\ 0 \end{pmatrix} \quad \boldsymbol{w} = \begin{pmatrix} -3 \\ 0 \\ -1 \end{pmatrix}$$

A) Which pair is orthogonal? Why?

B) Compute $\boldsymbol{v} \times \boldsymbol{w} - \boldsymbol{w} \times \boldsymbol{v}$

C) Compute the scalar triple product $\boldsymbol{v} \cdot (\boldsymbol{w} \times \boldsymbol{t})$

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PROBLEM 2: Consider the following differential form fields:

$$\alpha = z\,dx - x\,dy + 2y\,dz \qquad \& \qquad \beta = x^2\,dy \wedge dz - y^3\,dx \wedge dy$$

A) Compute and simplify $\alpha \wedge \beta$.

ANSWER

B) Compute and simplify the derivative $d\beta$.

ANSWER

C) Evaluate α at $x = 4, y = 3, z = 0$.

ANSWER

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PROBLEM 3: Consider the function $f(x, y, z) = xyz$ and the parametrized path γ given by:

$$\gamma(t) = \begin{pmatrix} 2t \\ 3t \\ t \end{pmatrix} \quad 0 \leq t \leq 1$$

A) What is the integral of f over γ?

ANSWER

B) What is the average of f over γ?

ANSWER

C) What is the integral of df over γ?

ANSWER

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PROBLEM 4: Consider the following matrix A:

$$A = \begin{bmatrix} -3 & 2 & 0 & 0 \\ -5 & 3 & 0 & 0 \\ 0 & 0 & 5 & 6 \\ 0 & 0 & 6 & 7 \end{bmatrix}$$

A) Compute the inverse of A by any means you wish. *(There is a blank page following this for extra work if you decide to do this the long way…)*

B) Solve the equation $A = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ for $\boldsymbol{x} = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$.

Hello…

This is a blank page for extra work on Problems 4 or 5, if you want it.

Please feel free to leave this blank, or to use it for testing out ideas or drawings…

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PROBLEM 5: Draw reasonable, labelled pictures of the following. *(If you have trouble drawing it, give a very clear written explanation…)*

A) The planar vector field $\vec{F} = -y\,\boldsymbol{i} + x\,\boldsymbol{j}$

B) The domain of integration of the integral

$$\int_{-2}^{2}\int_{0}^{\sqrt{4-y^2}}\int_{0}^{\sqrt{4-x^2-y^2}} x + 2y - 3z\;dz\;dx\;dy$$

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PROBLEM 6: Consider the vector field

$$\vec{F} = x\,\boldsymbol{i} + y\,\boldsymbol{j} + (x^2 + y^2)z\,\boldsymbol{k}$$

A) What is the flux 2-form associated to $\vec{F}$?

B) What is the divergence of $\vec{F}$?

C) What is the flux of $\vec{F}$ out of the closed cylinder $x^2 + y^2 = 9$ with $-2 \leq z \leq 2$ (including the endcaps; use the “outward-pointing-normal” orientation).

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PROBLEM 7: Consider the function $f(x, y) = xy + xy^2 - x^2y$.
A) Assuming that f has critical points at $(0,0)$ and $(1,0)$, classify them.

B) If you constrain the function f above to the ellipse given by

$$4x^2 + 9y^2 = 36$$

then the critical points are totally different (and harder to find!). ***Set up*** the Lagrange equations for this problem: be explicit *(don't simply write down the generic statement of the Lagrange formula)*.

C) Do you want to solve these equations for the critical points? (HINT: say NO!)

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PROBLEM 8: Evaluate the following integral:

$$\int_{y=0}^{3} \int_{x=\frac{y^2}{2}}^{\frac{y^2}{2}+2} y^2\sqrt{(2x-y^2)^3}\, dx\, dy$$

ANSWER

by using the change of coordinates

$$u = 2x - y^2$$
$$v = y$$

You must show all steps and work carefully to get full credit.

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PROBLEM 9: Compute the flux of the curl of the vector field

$$\vec{F} = y\, e^{z^2}\, \boldsymbol{i} - x \cos z^2\, \boldsymbol{j} + e^{3y} z^5\, \boldsymbol{k}$$

through the surface given by $z = 9 - x^2 - y^2$ with $z \geq 0$, oriented by the z-axis.
(Wow, that's kinda scary-looking… Hmmmm…. Let's think…)

ANSWER

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PROBLEM 10: Consider the parametrized surface in 3-D given by

$$G\begin{pmatrix} s \\ t \end{pmatrix} = \begin{pmatrix} s+t \\ s^2 - t^2 \\ st \end{pmatrix} \quad : \quad 0 \leq s, t \leq 1$$

A) Compute the derivative $[DG]$.

B) Tell me a pair of vectors that are tangent to the surface at the point

$$G\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$$

(No work needed if you did part A right…)

C) Set up and compute the integral of the 2-form $\beta = z\,dx \wedge dy$ over the surface.
(Note the simple bounds on the parameters: $0 \leq s \leq 1$ *and* $0 \leq t \leq 1$*)*

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PROBLEM 11: Short answers / fill in; be precise for partial credit.

A) Green's Theorem says that *(under the correct hypotheses…)*

$$\int_{\partial D} F(x,y)dx + G(x,y)dy \ =$$

B) Let $f\colon \mathbb{R}^3 \to \mathbb{R}^3$ and $g\colon \mathbb{R}^3 \to \mathbb{R}^3$ be inverse functions with $f(\mathbf{0}) = g(\mathbf{0}) = \mathbf{0}$. Then, it follows that

$$[Df]_{\mathbf{0}}[Dg]_{\mathbf{0}} =$$

C) The volume element for 3-D spherical coordinates equals… *[give the formula]*

D) The generalized Stokes' Theorem for differential forms states that, for α a differentiable k-form field on a suitable domain D… *[fill in the formula please]*

E) What is the definition of the level set $f^{-1}(c)$ of a function $f\colon \mathbb{R}^n \to \mathbb{R}$?

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PROBLEM 12: Consider the following joint probability density function for random variables X and Y on the domain $1 \le x < \infty$ and $1 \le y < \infty$ in $\mathbb{R}^2$:

$$\rho = 2x^{-2}y^{-3}$$

A) What is the probability that $X \le 2$ and $Y \le 3$?

ANSWER

B) Compute the expectation $\mathbb{E}(Y)$ (that is, the "mean" of the Y variable).

ANSWER

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PROBLEM 13:

A) Compute the moment of inertia of the solid unit-density tube given by

$$1 \leq x^2 + y^2 \leq 4 \quad \& \quad -5 \leq z \leq 5$$

ANSWER

about the z-axis.

B) If you compute the inertia matrix $[I]$ of an object centered at the origin to be

$$[I] = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 4 & -1 \\ 0 & -1 & 3 \end{bmatrix}$$

Then what is the moment of inertia of this object about the diagonal $x = y = z$?

ANSWER

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PROBLEM 14: Consider the function

$$f(x, y, z) = e^{xy}(3x - z)\cos(yz)$$

A) Taylor expand f about the origin, including only terms of ***degree 4 and lower***.

B) What is $[Df]$ at the origin?

C) At the origin, to which input is the value of f *most sensitive*? *Least sensitive*? No explanation needed, unless you want to…